Angular Momentum

[Reading: Shankar chpt. 12, Griffiths chpt. 4]

Orbital angular momentum

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

$$\boxed{\begin{aligned} \hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{aligned}}$$

Verify: all these are hermitian

Fundamental commutation relations

Using $[\hat{x}, \hat{p}_x] = i\hbar \hat{I}$, etc. along with the commutator identity $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$, one can check that

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y}$$

So the 3 components of angular momentum don't commute!

But, can check that $\hat{L}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

does commute with each of the three components!

$$[\hat{L}^2, \hat{L}_x] = 0, \quad [\hat{L}^2, \hat{L}_y] = 0, \quad [\hat{L}^2, \hat{L}_z] = 0$$

Implication:

We can't measure $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ simultaneously (since they don't commute), but can measure $\hat{L}^2$ together with any one of the three components (we'll choose $\hat{L}_z$ for convenience).

$\Rightarrow$ $\hat{L}^2$ and $\hat{L}_z$ are compatible observables and thus share a common set of eigenvectors.

Other types of angular momentum

Classically, particles only have orbital angular momentum (denoted $\hat{\vec{L}}$), but quantum mechanically, they can also have an internal angular momentum called spin (denoted $\hat{S}$). When talking about angular momentum in general, we'll use the symbol $\hat{J}$. We assume the basic commutation relations hold:

$$\boxed{[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y}$$

Similarly, $\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ commutes with each component $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$.

→ $\hat{J}^2$ and $\hat{J}_z$ are compatible
(have common set of eigenvectors

<u>Using ladder operators to find the eigenvalues of</u> $\hat{J}^2$, $\hat{J}_z$:

Let $|\alpha, \beta\rangle$ denote the (common) eigenvectors of $\hat{J}^2$ and $\hat{J}_z$:

$$\hat{J}^2 |\alpha,\beta\rangle = \alpha |\alpha,\beta\rangle, \quad \hat{J}_z |\alpha,\beta\rangle = \beta |\alpha,\beta\rangle$$

Can define raising and lowering operators to help us determine α and β (analogous to what we did for the harmonic oscillator).

Define $\boxed{\begin{aligned} \hat{J}_+ &\equiv \hat{J}_x + i\hat{J}_y \\ \hat{J}_- &\equiv \hat{J}_x - i\hat{J}_y \end{aligned}}$ $\leftarrow$ <u>not</u> hermitian

Easy to verify that

$$[\hat{J}^2, \hat{J}_\pm] = 0 \,, \quad [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm \,, \quad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z$$

Tedious (but not hard) to also verify

$$\hat{J}^2 = \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z$$
$$\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z$$

We now want to consider how $\hat{J}_\pm$ affect the eigenstate $|\alpha, \beta\rangle$

Observe:

$$\hat{J}^2 \hat{J}_+ |\alpha,\beta\rangle = \hat{J}_+ \hat{J}^2 |\alpha,\beta\rangle = \alpha \hat{J}_+ |\alpha,\beta\rangle$$

So $\hat{J}_+ |\alpha,\beta\rangle$ is an eigenstate of $\hat{J}^2$, and it has the same eigenvalue as state $|\alpha,\beta\rangle$

Observe:

$$\hat{J}_z \hat{J}_+ |\alpha,\beta\rangle = [\hat{J}_+ \hat{J}_z + \hbar \hat{J}_+] |\alpha,\beta\rangle$$
$$= (\beta + \hbar) \hat{J}_+ |\alpha,\beta\rangle$$

So $\hat{J}_+ |\alpha,\beta\rangle$ is an eigenstate of $\hat{J}_z$ with eigenvalue $\beta + \hbar$

<u>Summary</u>: $\hat{J}_+$ is a raising operator

$$\boxed{\hat{J}_+ |\alpha,\beta\rangle = C^+_{\alpha,\beta} |\alpha,\beta+\hbar\rangle}$$

a constant to be determined

Likewise, $\hat{J}_-$ is a lowering operator

$$\hat{J}_- |\alpha, \beta\rangle = c^-_{\alpha\beta} |\alpha, \beta - \hbar\rangle .$$

There is a limit to the number of times that we can apply the raising (or lowering) operators (see sect. 12.5 of your text). In other words, there is a maximum and minimum value that β can take on:

$$\beta^2 \le \alpha$$

interpretation: the z-component of angular momentum cannot exceed the angular momentum itself.

A little more work shows that

$$\alpha = \hbar^2 j(j+1) \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\beta = -j\hbar, -j\hbar + \hbar, \ldots j\hbar$$

Cleaning up the notation a bit:

eigenstates $|j,m\rangle$

$$\boxed{\begin{aligned} &\hat{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle \\ &\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle \\ &\text{where } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots \\ &\qquad m = -j, -j+1, -j+2, \ldots, j \end{aligned}}$$

total squared angular momentum and z-component of angular momentum are discrete

Note: can also verify that

$$\langle j', m' | j, m \rangle = \delta_{j'j} \delta_{m'm}$$

The action of the raising and lowering operators is

$$\hat{J}_+ \, |j,m\rangle = \hbar \sqrt{j(j+1) - m(m+1)} \, |j,m+1\rangle$$

$$\hat{J}_- \, |j,m\rangle = \hbar \sqrt{j(j+1) - m(m-1)} \, |j,m-1\rangle$$

(Question: Given the above, how would you calculate $\hat{J}_x \, |j,m\rangle$

Hint: Just express $\hat{J}_x$ in terms of the raising and lowering operators!)

The matrix representation of angular momentum:

All of the operators considered above (e.g., $\hat{J}^2, \hat{J}_z, \hat{J}_\pm, \hat{J}_x, \hat{J}_y, \hat{J}_z$) can be represented as matrices in the $|j,m\rangle$ eigenbasis. In your homework set you will explore how to do this.

More on orbital angular momentum:

The previous considerations hold, but we'll change notation slightly

e.g. eigenstates of $\hat{L}^2$, $\hat{L}_z$ are

$$|l, m\rangle$$

$$\left(\begin{aligned} \hat{L}^2 |l, m\rangle &= \hbar^2 l(l+1) |l, m\rangle \\ \hat{L}_z |l, m\rangle &= \hbar m |l, m\rangle \end{aligned} \right)$$

Orbital angular momentum in spherical coordinate representation:

Recall: $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$, etc.

In cartesian coordinate representation,

$\hat{p}_z \rightarrow \frac{\hbar}{i} \partial_z$, $\hat{p}_y \rightarrow \frac{\hbar}{i} \partial_y$, etc.

So, for example,

$$\hat{L}_z = -i\hbar\left(y\partial_z - z\partial_y\right)$$

Converting to spherical coordinates yields

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}}$$

Similarly, after a bit of work we find the spherical representations of the other operators:

$$\boxed{\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}$$

Likewise,

$$\boxed{\begin{aligned} L_+ &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) \\ L_- &= \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) \end{aligned}}$$

Now we're ready to find the orbital angular momentum eigenstates in the spherical-coordinate representation:

$$|\ell, m\rangle \xrightarrow{\text{spherical coordinates}} Y_\ell^m(\theta,\phi) \equiv \langle\theta,\phi|\ell,m\rangle$$

"spherical harmonics"

Let's start with the operator $\hat{L}_z$

Since $\hat{L}_z|\ell,m\rangle = m\hbar|\ell,m\rangle$

$$\Rightarrow \quad -i\hbar\frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = m\hbar\, Y_\ell^m(\theta,\phi)$$

Let's guess a solution of the form $Y_\ell^m(\theta,\phi) = f(\theta)\,g(\phi)$

$$\Rightarrow \quad -i\hbar \frac{\partial}{\partial\phi} g(\phi) = \hbar m\, g(\phi)$$

$$\Rightarrow \quad g(\phi) = e^{im\phi}$$

Important point: If we rotate in physical space by $\phi \to \phi + 2\pi$, it had better be the case that we get back the same wavefunction: $g(\phi+2\pi) = g(\phi)$.
This in turn implies that

$$e^{i2\pi m} = 1$$

$$\longrightarrow \quad m = 0, \pm 1, \pm 2, \ldots$$

But from earlier considerations (about general angular momentum) we already determined that

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$m = -j, -j+1, \ldots, j-1, j$$

<u>Observe</u>: for orbital angular momentum we found that m must be an integer, but for general angular momentum we found that it could be a half-integer. Hence, the condition on orbital angular momentum is more restrictive! The only way to avoid conflict is if, for orbital angular momentum, the allowed values of ℓ (i.e., j) are strictly integer.

In summary, for <u>orbital angular momentum</u>,

$$\ell = 0, 1, 2, \ldots$$

$$m = -\ell, -\ell+1, \ldots, \ell$$

(Terminology: ℓ is often called the "orbital angular momentum quantum number" and m the "magnetic quantum number.")

In any event, we have found that the eigenstates of $\hat{L}^2, \hat{L}_z$ are of the form

$$Y_\ell^m(\theta,\phi) = f(\theta)\, g(\phi) \quad \text{with } g(\phi) = e^{im\phi}$$

Now let's find the exact form of the $f(\theta)$ part:

Since $$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

The eigenvalue equation $\hat{L}^2 Y_\ell^m(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\phi)$ becomes

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)\right]f_{\ell m}(\theta) = 0$$

here I've called $f(\theta) \longrightarrow f_{\ell m}(\theta)$ to make the ℓ, m - dependence of f explicit

This equation is known as the *Legendre differential equation*. If it looks somewhat familiar to you, it's because if you set m=0 you get Laplace's equation, which you studied in electromagnetism! (In that case, the solutions are the Legendre polynomials).

Now, the solutions to the above equation are also well known. They are called the *associated Legendre functions*, denoted $P_\ell^m(\cos\theta)$

Here are the first few solutions:

$$P_1^1(\cos\theta) = \sin\theta \qquad P_1^0(\cos\theta) = \cos\theta \qquad P_2^2(\cos\theta) = 3\sin^2\theta$$

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta \qquad P_2^0(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3^3(\cos\theta) = 15\sin\theta(1-\cos^2\theta) \qquad P_3^2(\cos\theta) = 15\sin^2\theta\cos\theta$$

$$P_3^1(\cos\theta) = \frac{3}{2}\sin\theta(5\cos^2\theta - 1) \qquad P_3^0(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

[Note: $P_\ell^{-m}(\cos\theta) = P_\ell^m(\cos\theta)$]

Various recursion relations exist which enable you to generate these functions (see, e.g., Griffiths). I'll show you an alternate method for doing this in a little bit.

For now, let's summarize what we've found thus far:

The eigenvalues of the operators $\hat{L}^2, \hat{L}_z$ are (respectively):

$$\hbar^2 \ell(\ell+1) \qquad \text{where} \qquad \ell = 0, 1, 2, \ldots$$

$$\hbar m \qquad \text{where} \qquad m = -\ell, -\ell+1, \ldots, \ell$$

These two operators share common eigenvectors, denoted $|\ell, m\rangle$

They obey the orthogonality relations $\langle \ell, m | \ell', m' \rangle = \delta_{\ell\ell'} \delta_{mm'}$

In the (spherical-) coordinate representation, these eigenvectors are called the **spherical harmonics**, and, in normalized form, are given by

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} \, P_\ell^m(\cos\theta) e^{im\phi} \qquad m \geq 0$$

[For negative values of m, use $Y_\ell^{-m} = (-1)^m \left(Y_\ell^m\right)^*$]

Note that all the messy constants out front have been determined from the normalization condition

$$\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \left| Y_\ell^m(\theta,\phi) \right|^2 \sin\theta \, d\theta \, d\phi = 1$$

Before describing their properties and showing you what they look like, let me show you (as promised) an alternate way of deriving the spherical harmonics. We'll do this via the ladder operators:

Recall: state $|\ell, m\rangle$ $\quad m_{max} = \ell$

$$\Rightarrow \quad \hat{L}_+ |\ell, \ell\rangle = 0 \quad \text{(since can't raise } m \text{ above } \ell\text{)}$$

$$\Rightarrow \quad \hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right) Y_\ell^\ell(\theta,\phi) = 0$$

Let $Y_\ell^\ell(\theta,\phi) = f_{\ell\ell}(\theta) e^{i\ell\phi}$

$$\Rightarrow \quad \frac{\partial}{\partial\theta} f_{\ell\ell}(\theta) = \ell \cot\theta$$

$$\Rightarrow \quad f_{\ell\ell}(\theta) = \sin^{\ell}(\theta)$$

So the (normalized) solution is

$$Y_{\ell}^{\ell}(\theta,\phi) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^{\ell}}{2^{\ell}\ell!} \sqrt{\frac{(2\ell+1)!}{2}}\, e^{i\ell\phi} \sin^{\ell}(\theta)$$

Now just hit this repeatedly with $\hat{L}_-$ to generate the lower m states!

A little algebra gives

$$Y_{\ell}^{m}(\theta,\phi) = \frac{(-1)^{\ell}}{2^{\ell}\ell!} \sqrt{\left(\frac{2\ell+1}{4\pi}\right)\frac{(\ell+m)!}{(\ell-m)!}}\, e^{im\phi} \frac{1}{\sin^{m}(\theta)} \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}} (\sin\theta)^{2\ell}$$

(I never said it was pretty!)

Visualizing the spherical harmonics

The best way is to use one of the many applets available on the web. I will give you the address of a good site shortly.

Rotational symmetry and Angular momentum

In our earlier discussion of symmetries, you may have noticed that we didn't examine one of the most basic symmetries in nature – rotational symmetry. This delay was intentional. Now we take up this important issue

In the (spherical-) coordinate representation, an infinitesimal rotation of a (spinless) particle by angle $d\phi$ about the z-axis is described by

$$\hat{R}_z(d\phi)\ \psi(r,\theta,\phi) = \psi(r,\theta,\phi - d\phi)$$

Taylor expanding the RHS gives

$$\psi(r,\theta,\phi - d\phi) \simeq \psi(r,\theta,\phi) - d\phi\ \frac{\partial}{\partial\phi}\psi(r,\theta,\phi)$$

Hence, $$\hat{R}_z(d\phi) = I - d\phi \frac{\partial}{\partial \phi}$$

Bur recall that $$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\Rightarrow \quad \hat{R}_z(d\phi) = I - \frac{i}{\hbar} d\phi \, \hat{L}_z$$

So the z-component of angular momentum is the generator of rotations about the z-axis! (Analogous results hold for the x- and y- components as well.)

Generalizing this relation to rotations about an arbitrary axis specified by unit vector $\hat{n}$, we have

$$\hat{R}(d\phi) = I - \frac{i}{\hbar} d\phi \, \hat{n} \cdot \hat{\vec{L}}$$

$\Rightarrow$ $\boxed{\text{Orbital angular momentum is the generator of spatial rotations!}}$

Finite rotations:

Revisiting earlier symmetry arguments, you should not be surprised to find that rotation by a finite angle is described by operator

$$\hat{R}_n(\phi) = e^{-\frac{i}{\hbar}\phi\,\hat{n}\cdot\hat{\vec{L}}}$$

[For systems with spin, the $\hat{\vec{L}}$ appearing above should be replaced by the total angular momentum $\hat{\vec{J}}$ (more on this later).]

Rotational invariance

A system is rotationally invariant if its hamiltonian commutes with the generator of rotations, i.e., angular momentum:

$$\text{if} \quad [\hat{H}, \hat{\vec{L}}] = 0 \quad \longrightarrow \quad \text{rotation symmetry}$$

This will a very common occurrence: Assuming space is *isotropic*, then rotating an isolated physical system can't change it's behavior. So we expect it to be rotationally symmetric.

One of the most common examples of a rotationally symmetric system is a particle in a central potential, where $\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$.

And since rotational symmetry implies that the generator commutes with the hamiltonian, then we know (by Ehrenfest's theorem) that angular momentum is conserved!

Hence, we can say (rather deeply) that the isotropy of space is the reason why we have the law of conservation of momentum!

A lot more can be said about rotation operators (e.g., Euler angles, the Wigner functions, etc.) We'll stop here for now.